Reciprocity in potential scattering and anti-pseudo-unitarity of the fundamental transfer matrix

Ali Mostafazadeh
Koç University, Istanbul

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Joint Work with

Farhang Loran

(Isfahan Univ. Tech., Iran)

Outline:

- Potential scattering, transfer matrix, and Reciprocity Theorem in 1D
- Dynamical formulation of stationary scattering in 1D and Reciprocity Theorem
- Potential scattering & Reciprocity Theorem in 3D
- Dynamical formulation of stationary scattering and fundamental transfer matrix in 3D
- Anti-pseudo-unitarity of the fundamental transfer matrix and a proof of Reciprocity Theorem
- Concluding remarks

Scattering in 1D $\Psi(x,t) = e^{-i\omega t}\psi(x)$

- Time-Indep. Schrödinger Eq.: $-\psi(x)'' + v(x)\psi(x) = k^2\psi(x)$
- $v : \mathbb{R} \to \mathbb{C}$ is a possibly k-dependent **short-range potential**.

$$x\, v(x) \to 0 \quad \text{for} \quad x \to \pm\infty$$

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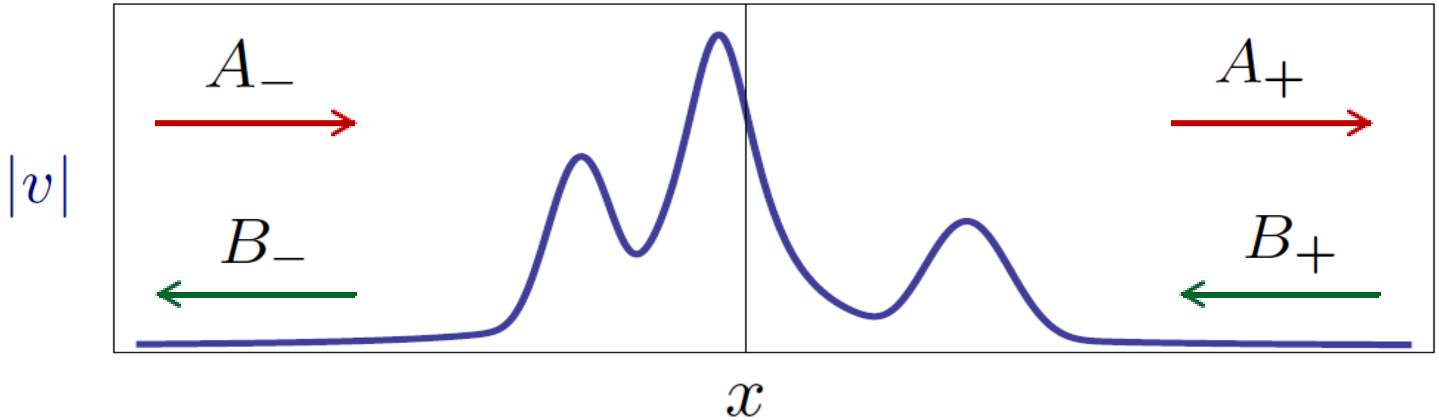
$$\psi(x) \to \begin{cases} A_- e^{ikx} + B_- e^{-ikx} & \text{for } x \to -\infty \\ A_+ e^{ikx} + B_+ e^{-ikx} & \text{for } x \to +\infty \end{cases}$$



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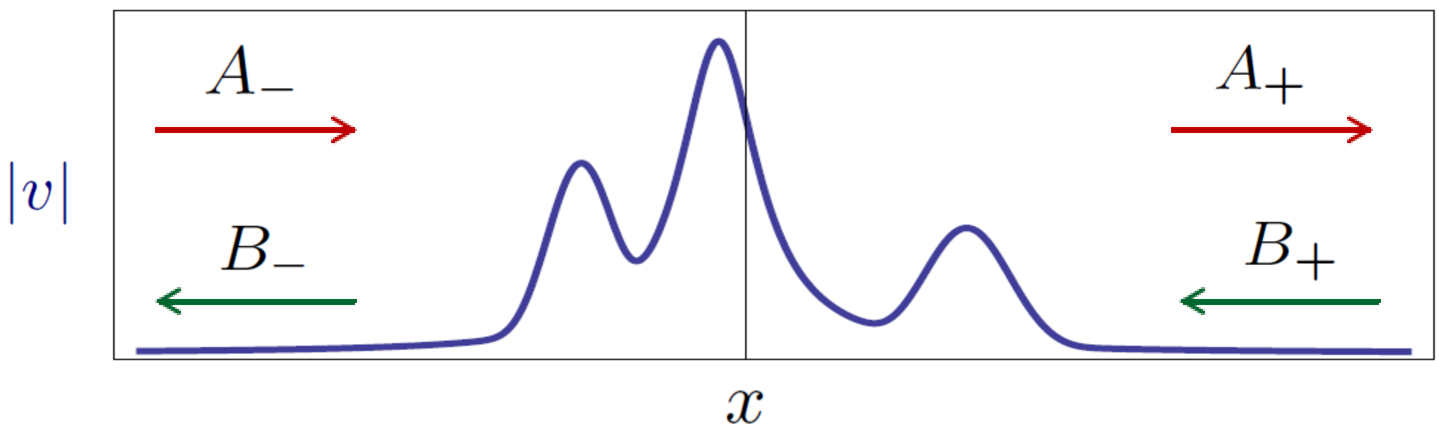
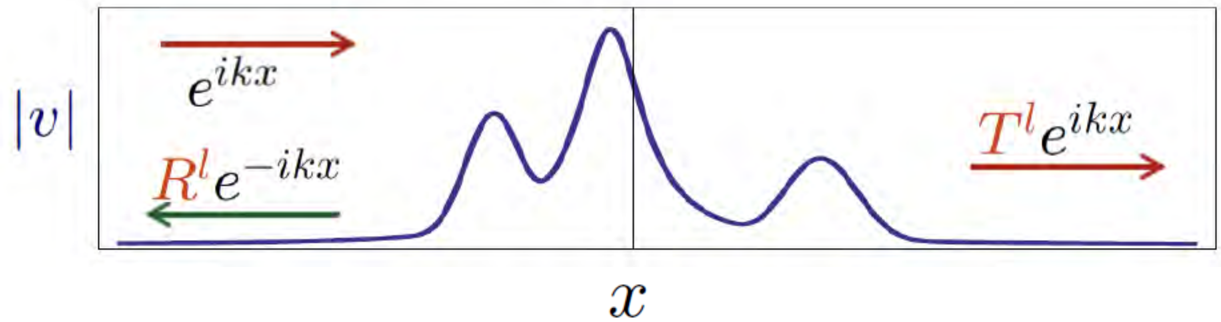


- Transfer matrix: $\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \mathbf{M} \begin{bmatrix} A_- \\ B_- \end{bmatrix}.$

- Scattering from the left and right:

$$\psi^{\text{left}}(x) \to N^l \times \begin{cases} e^{ikx} + R^l\, e^{-ikx} & \text{for} \quad x \to -\infty \\ T^l\, e^{ikx} & \text{for} \quad x \to +\infty \end{cases}$$



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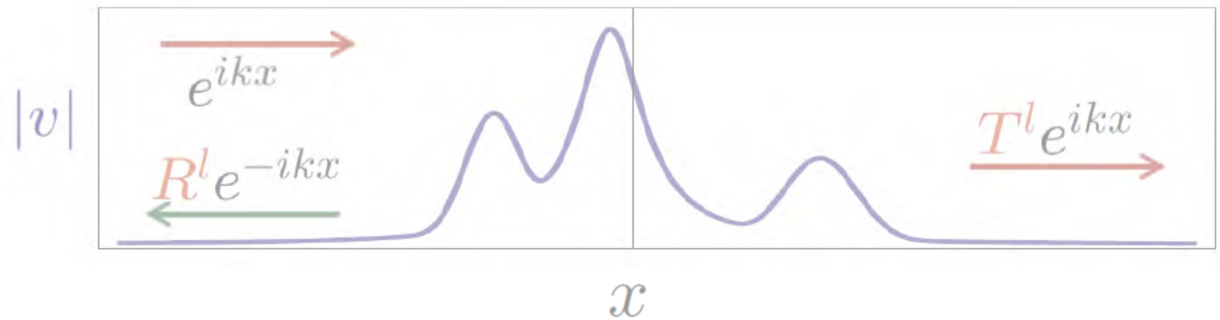

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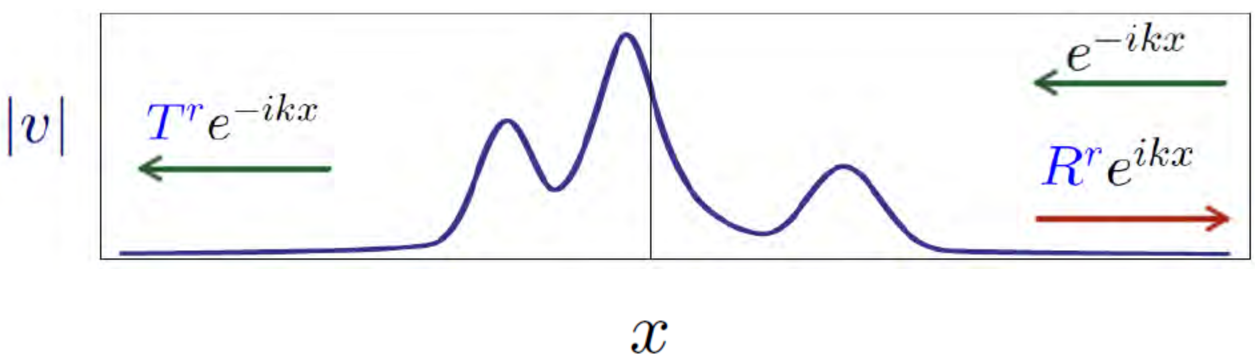

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$$\boxed{R^l = -\frac{M_{21}}{M_{22}}, \quad T^l = \frac{\det \mathbf{M}}{M_{22}}, \quad R^r = \frac{M_{12}}{M_{22}}, \quad T^r = \frac{1}{M_{22}},}$$

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$$T^l = \frac{\det \mathbf{M}}{M_{22}}, \quad T^r = \frac{1}{M_{22}} \quad \Rightarrow$$

$$T^l = T^r \quad \Leftrightarrow \quad \det \mathbf{M} = 1$$

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$$\Leftrightarrow \quad \mathbf{M} \in Sl(2, \mathbb{C})$$

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$$\begin{aligned} T^l = T^r \quad &\Leftrightarrow \quad \det \mathbf{M} = 1 \\ &\Leftrightarrow \quad \mathbf{M} \in Sl(2,\mathbb{C}) = Sp(2,\mathbb{C}) \\ &\Leftrightarrow \quad \mathbf{M}^T\, \boldsymbol{\Omega}\, \mathbf{M} = \boldsymbol{\Omega} \end{aligned}$$

$$\boldsymbol{\Omega} := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\mathbb{C}^{m\times n}$: Vector space of complex $m \times n$ matrices

$\mathcal{T} : \mathbb{C}^{2\times 1} \to \mathbb{C}^{2\times 1}$ be complex-conjugation: $\mathcal{T}\mathbf{v} = \mathbf{v}^*$

$\Rightarrow$ $\mathcal{T}$ is an antiunitary involution: $\mathcal{T}^\dagger = \mathcal{T}^{-1} = \mathcal{T}$.

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Identify 2×2 matrices $\mathbf{A}$ with the linear operators:

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$\Rightarrow$ $\boxed{\text{Reciprocity is equivalent to the } \Omega\mathcal{T}\text{-anti-pseudo-unitarity of } \mathbf{M}.}$

Dynamical formulation of stationary scattering in 1D:

Theorem: Let $\mathbf{M}$ be the transfer matrix for a short-range potential $v:\mathbb{R}\to\mathbb{C}$,

$$\mathbf{H}(t) := \frac{v(t)}{2k}\begin{bmatrix} 1 & e^{-2ikt} \\ -e^{2ikt} & -1 \end{bmatrix},$$

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The "x" in $v(x)$ plays the role of "time". With $t\to x$,

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$$(-i)^3 \int_{x_0}^{x} dx_3 \int_{x_0}^{x_3} dx_2 \int_{x_0}^{x_2} dx_1 \mathbf{H}(x_3)\mathbf{H}(x_2)\mathbf{H}(x_1) + \cdots$$

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$$\Rightarrow \qquad \boxed{\mathbf{M} = \mathscr{T} \exp\left[-i\int_{-\infty}^{\infty} dx\, \mathbf{H}(x)\right]}$$

$$\mathrm{tr}[\mathbf{H}(x)] = 0 \;\Rightarrow\; \det \mathbf{M} = 1 \;\Leftrightarrow\; \text{Reciprocity}$$

$$T^l = \frac{\det \mathbf{M}}{M_{22}}, \qquad T^r = \frac{1}{M_{22}}$$

Scattering in 3D:

- Time-indep. Schrödinger Eq.: $[-\nabla^2 + v(\mathbf{r})]\psi(\mathbf{r}) = k^2\psi(\mathbf{r})$
- $v : \mathbb{R}^3 \to \mathbb{C}$ is a possibly k-dependent short-range potential.

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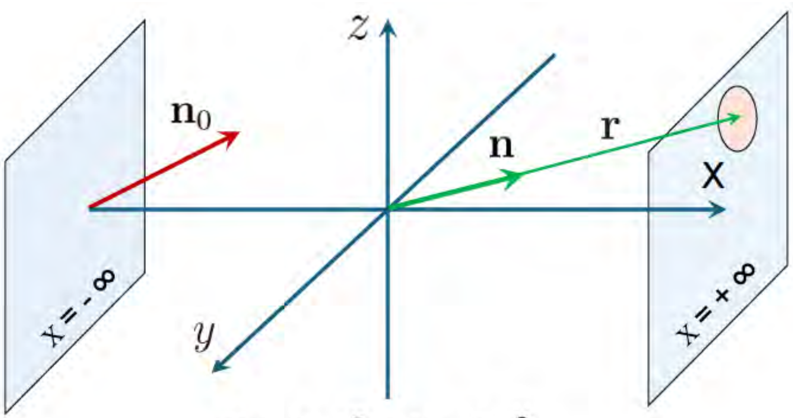
$$\mathbf{n}_0 := \frac{\mathbf{k}_0}{k}, \qquad \mathbf{n} := \frac{\mathbf{r}}{r}$$



$n_{0x} > 0$, $n_x > 0$

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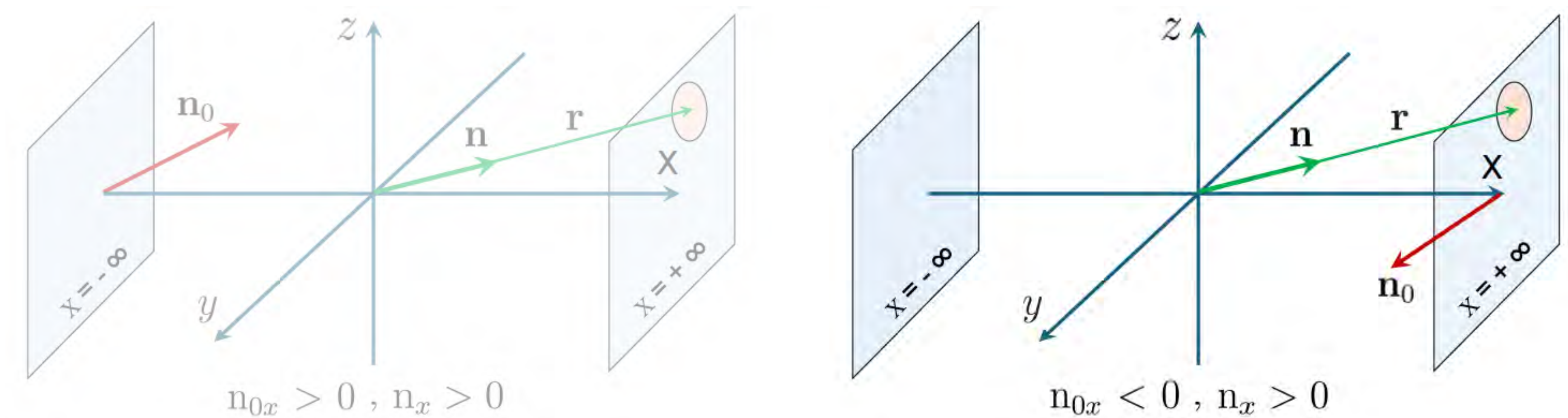

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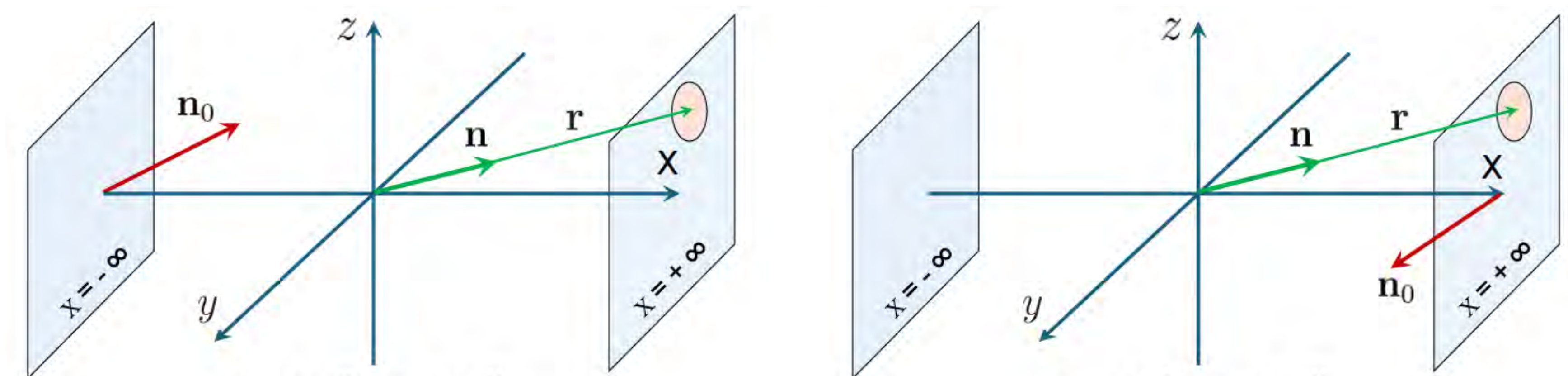
Reciprocity Theorem (3D): Let $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential. Then $\mathfrak{f}(\mathbf{n}_0, \mathbf{n}) = \mathfrak{f}(-\mathbf{n}, -\mathbf{n}_0)$.



$\mathrm{n}_{0x} > 0$, $\mathrm{n}_x > 0$ $\qquad$ $\mathrm{n}_{0x} < 0$, $\mathrm{n}_x > 0$

Reflection and transmission amplitudes in 3D:

Notation: For all $\mathbf{v} = (v_x, v_y, v_z) \in \mathbb{R}^3$, we define

$$\vec{v} := (v_y, v_z) \in \mathbb{R}^2$$

Examples: $\mathbf{r} = (x, y, z) \;\Rightarrow\; \vec{r} := (y, z)$

$$\mathbf{p} = (p_x, p_y, p_z) \;\Rightarrow\; \vec{p} := (p_y, p_z)$$

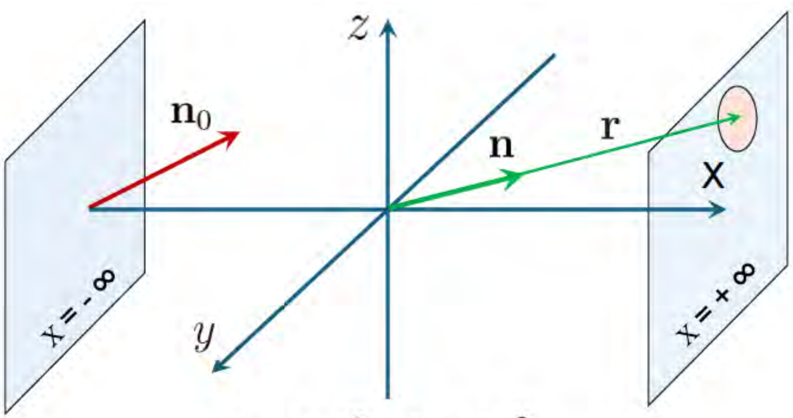
$$\mathbf{n}_0 = (\mathsf{n}_{0x}, \mathsf{n}_{0y}, \mathsf{n}_{0z}) \;\Rightarrow\; \vec{\mathsf{n}}_0 := (\mathsf{n}_{0y}, \mathsf{n}_{0z})$$

Reflection and transmission amplitudes in 3D:

Scattering solutions: $\psi(\mathbf{r}) \to \frac{N}{(2\pi)^{\frac{3}{2}}}\left[e^{i\mathbf{k}_0\cdot\mathbf{r}} + \frac{e^{ikr}}{r}\mathfrak{f}(\mathbf{n}_0,\mathbf{n})\right]$ for $r\to\infty$

$$\boxed{\begin{array}{ll} R^l(\mathbf{n}_0,\mathbf{n}) := 2\pi i k\,\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) & \text{for}\quad \mathrm{n}_{0x} > 0 > \mathrm{n}_x \\ T^l(\mathbf{n}_0,\mathbf{n}) := 2\pi i k\left[\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) - \Delta(\vec{\mathrm{n}}_0,\vec{\mathrm{n}})\right] & \text{for}\quad \mathrm{n}_{0x},\mathrm{n}_x > 0 \\ R^r(\mathbf{n}_0,\mathbf{n}) := 2\pi i k\,\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) & \text{for}\quad \mathrm{n}_{0x} < 0 < \mathrm{n}_x \\ T^r(\mathbf{n}_0,\mathbf{n}) := 2\pi i k\left[\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) - \Delta(\vec{\mathrm{n}}_0,\vec{\mathrm{n}})\right] & \text{for}\quad \mathrm{n}_{0x},\mathrm{n}_x < 0 \end{array}}$$

$$\Delta(\vec{\mathrm{n}}_0,\vec{\mathrm{n}}) := 2\pi i\,k^{-1}|\mathrm{n}_{0x}|\,\delta^2(\vec{\mathrm{n}}-\vec{\mathrm{n}}_0)$$



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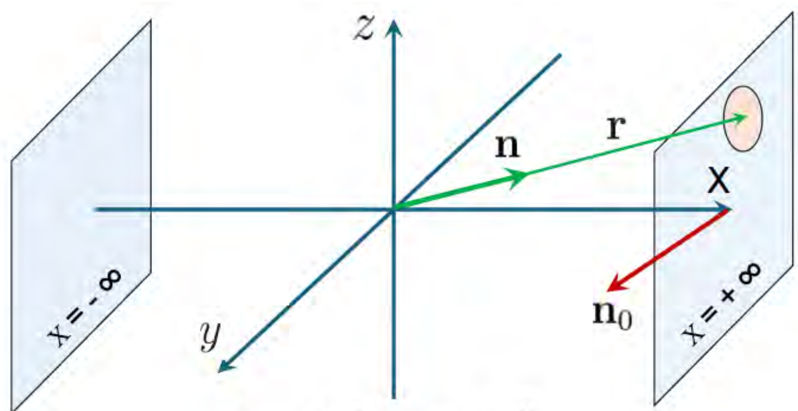

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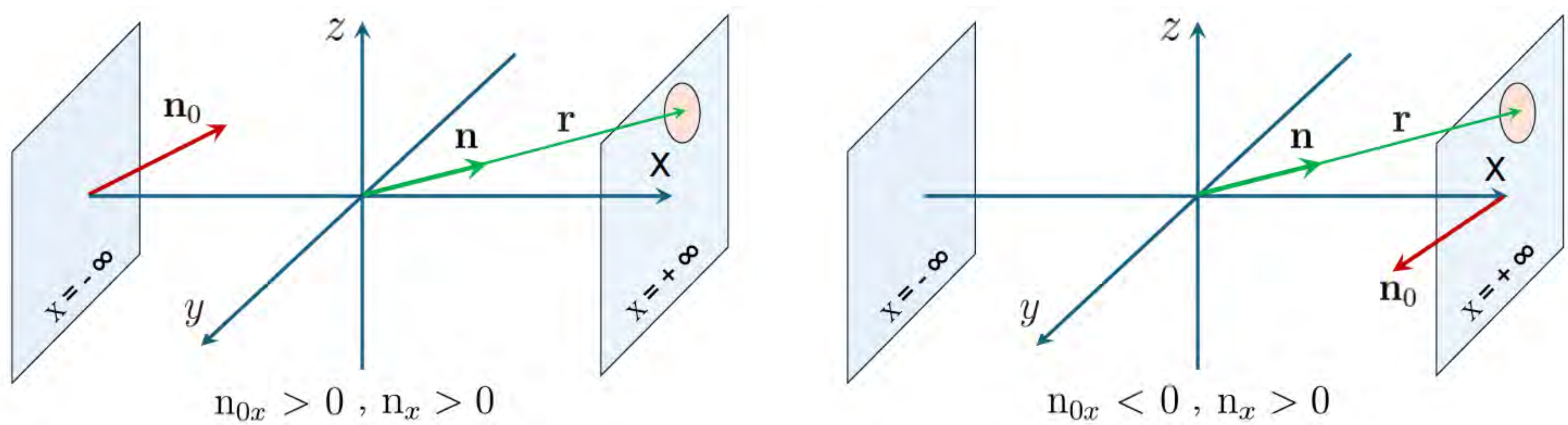
Reflection and transmission amplitudes in 3D:

Reciprocity $\Leftrightarrow$ $\mathfrak{f}(\mathbf{n}_0, \mathbf{n}) = \mathfrak{f}(-\mathbf{n}, -\mathbf{n}_0)$

$$\Leftrightarrow \begin{cases} R^l(\mathbf{n}_0, \mathbf{n}) = R^l(-\mathbf{n}, -\mathbf{n}_0) \\ R^r(\mathbf{n}_0, \mathbf{n}) = R^r(-\mathbf{n}, -\mathbf{n}_0) \\ T^l(\mathbf{n}_0, \mathbf{n}) = T^r(-\mathbf{n}, -\mathbf{n}_0) \\ T^r(\mathbf{n}_0, \mathbf{n}) = T^l(-\mathbf{n}, -\mathbf{n}_0) \end{cases}$$



Dynamical formulation of stationary scattering in 3D:

$$\mathbf{r} = (x, y, z) \;\Rightarrow\; \vec{r} := (y, z), \qquad \mathbf{p} = (p_x, p_y, p_z) \;\Rightarrow\; \vec{p} := (p_y, p_z)$$

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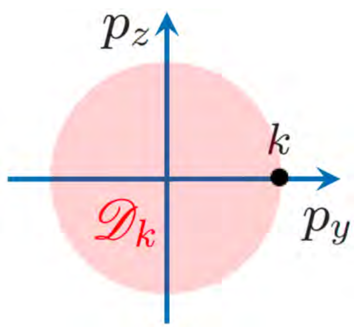

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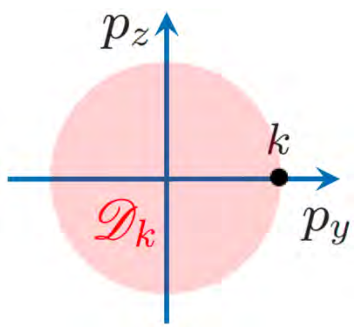

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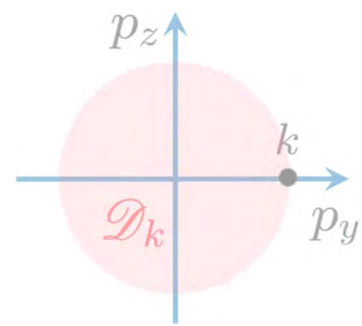

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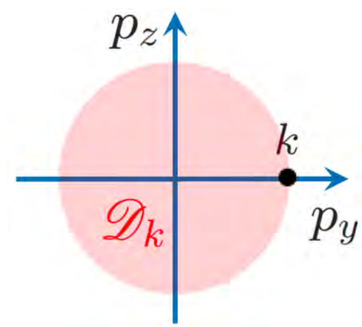

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For $x \to \pm\infty$,

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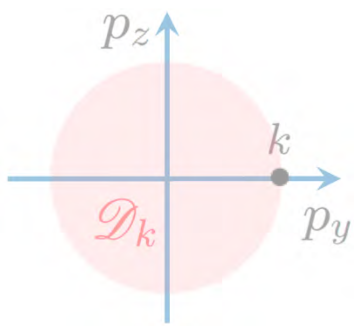

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$\mathscr{F}^m$: Space of functions $\mathbf{f} : \mathbb{R}^2 \to \mathbb{C}^{m\times 1}$

$$\mathscr{F}^m_k := \{ \mathbf{f} \in \mathscr{F}^m \,|\, \mathbf{f}(\vec{p}) = 0 \text{ for } \vec{p} \notin \mathscr{D}_k \}$$

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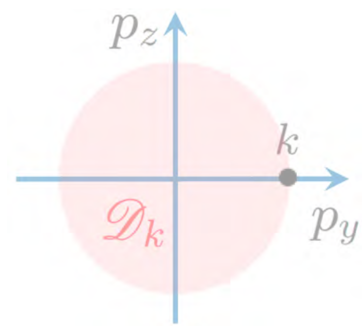

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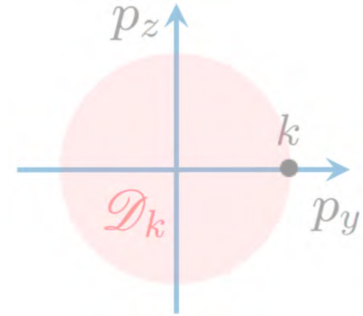

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$$A_\pm, B_\pm \in \mathscr{F}_k^1, \qquad \varpi(\vec{p}) := \sqrt{k^2 - \vec{p}^2}$$

Fundamental transfer matrix:

Linear operator $\widehat{\mathbf{M}}: \mathscr{F}_k^2 \to \mathscr{F}_k^2$ satisfying

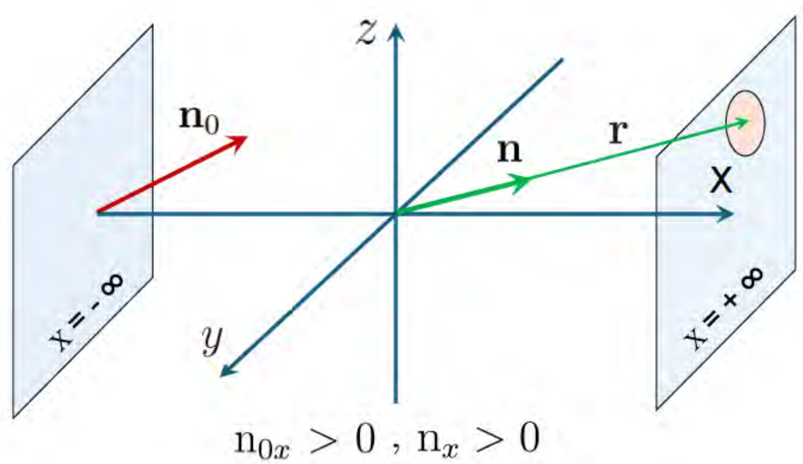
$$\boxed{\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \widehat{\mathbf{M}} \begin{bmatrix} A_- \\ B_- \end{bmatrix}}$$

$$\psi(\mathbf{r}) \to \frac{1}{(2\pi)^{7/2}} \int_{\mathscr{D}_k} \frac{d\vec{p}}{\varpi(\vec{p})}\, e^{i\vec{p}\cdot\vec{r}} \left[A_{\pm}(\vec{p})\, e^{i\varpi(\vec{p})x} + B_{\pm}(\vec{p})\, e^{-i\varpi(\vec{p})x}\right] \quad \text{for} \quad x \to \pm\infty$$

Scattering solutions: $\psi(\mathbf{r}) \to \dfrac{N}{(2\pi)^{\frac{3}{2}}} \left[e^{i\mathbf{k}_0\cdot\mathbf{r}} + \dfrac{e^{ikr}}{r}\, \mathfrak{f}(\mathbf{n}_0, \mathbf{n})\right] \quad$ for $\quad r \to \infty$

For $\mathsf{n}_{0x} > 0$: $A_{-}(\vec{p}) = 2\pi N\, \varpi(\vec{\mathsf{k}}_0)\, \delta^2(\vec{p} - \vec{\mathsf{k}}_0)$, $\qquad B_{+}(\vec{p}) = 0$

$$\vec{\mathsf{k}}_0 := k\, \vec{\mathsf{n}}_0$$



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For $\mathrm{n}_{0x} > 0$: $A_-(\vec{p}) = 2\pi N\, \varpi(\vec{\mathrm{k}}_0)\, \delta^2(\vec{p} - \vec{\mathrm{k}}_0)$, $\qquad B_+(\vec{p}) = 0$

For $\mathrm{n}_{0x} < 0$: $B_+(\vec{p}) = 2\pi N\, \varpi(\vec{\mathrm{k}}_0)\, \delta^2(\vec{p} - \vec{\mathrm{k}}_0)$, $\qquad A_-(\vec{p}) = 0$

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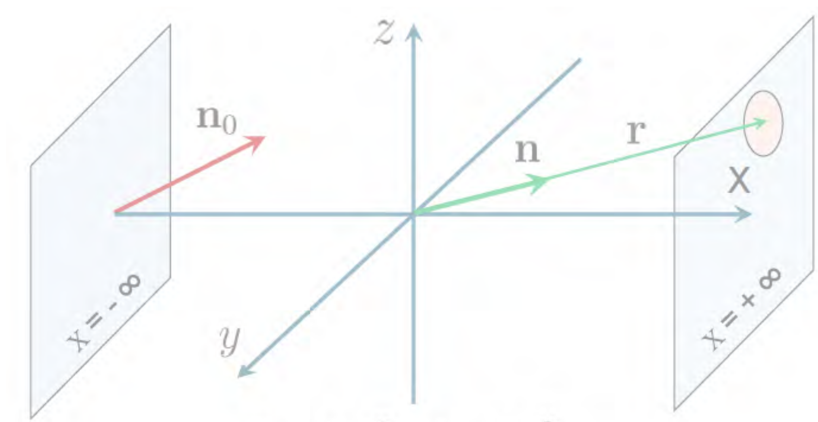

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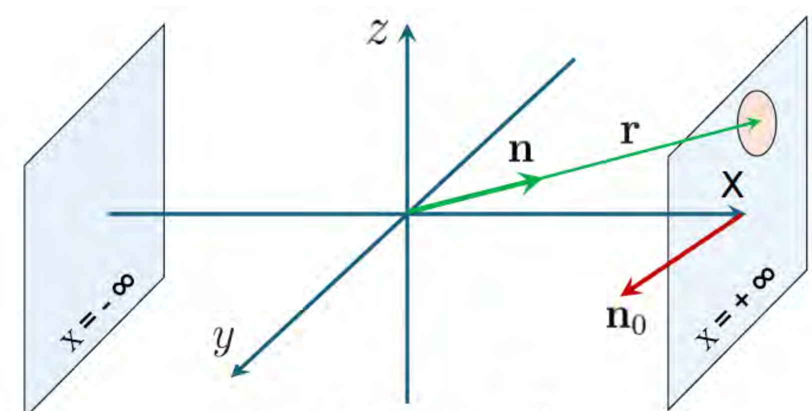

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For $n_{0x} > 0$: $A_-(\vec{p}) = 2\pi N\, \varpi(\vec{k}_0)\, \delta^2(\vec{p} - \vec{k}_0)$, $\qquad B_+(\vec{p}) = 0$

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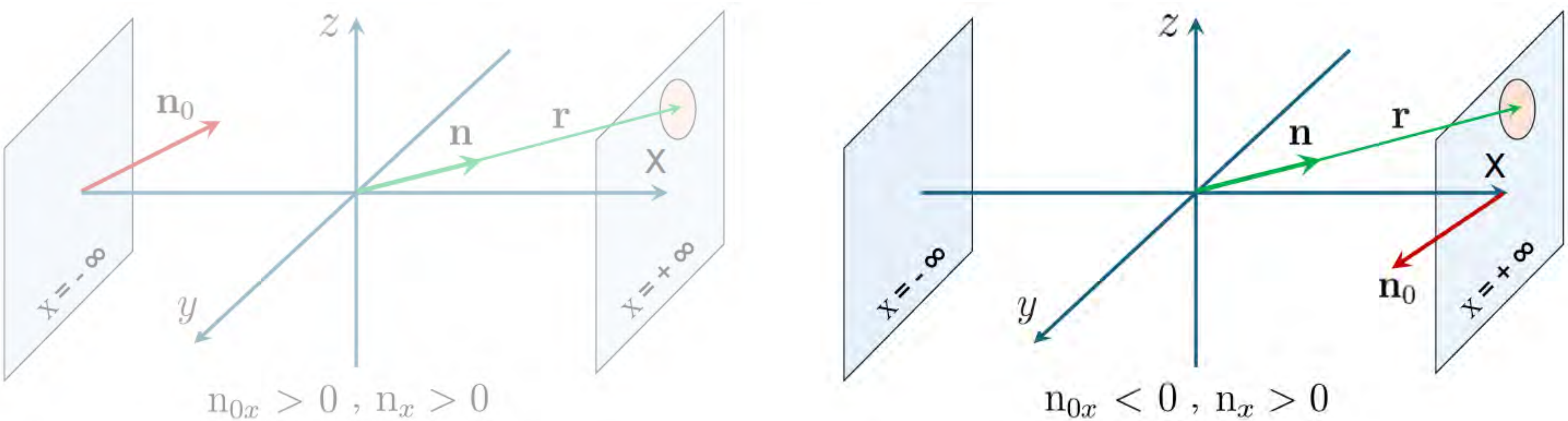
$\Rightarrow$ $\boxed{\mathfrak{f}(\mathbf{n}_0, \mathbf{n}) \text{ is given by } A_+ \text{ and } B_-.}$



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$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \widehat{\mathbf{M}} \begin{bmatrix} A_- \\ B_- \end{bmatrix}$$

Solve for A_+ and B_-.

$$\psi(\mathbf{r}) \to \frac{1}{(2\pi)^{7/2}} \int_{\mathscr{D}_k} \frac{d\vec{p}}{\varpi(\vec{p})}\, e^{i\vec{p}\cdot\vec{r}} \left[A_\pm(\vec{p})\, e^{i\varpi(\vec{p})x} + B_\pm(\vec{p})\, e^{-i\varpi(\vec{p})x}\right] \quad \text{for} \quad x \to \pm\infty$$

Scattering solutions: $\psi(\mathbf{r}) \to \frac{N}{(2\pi)^{\frac{3}{2}}} \left[e^{i\mathbf{k}_0\cdot\mathbf{r}} + \frac{e^{ikr}}{r}\mathfrak{f}(\mathbf{n}_0, \mathbf{n})\right] \quad \text{for} \quad r \to \infty$

For $\mathrm{n}_{0x} > 0$: $A_-(\vec{p}) = 2\pi N\,\varpi(\vec{\mathrm{k}}_0)\,\delta^2(\vec{p} - \vec{\mathrm{k}}_0)$, $\qquad B_+(\vec{p}) = 0$

For $\mathrm{n}_{0x} < 0$: $B_+(\vec{p}) = 2\pi N\,\varpi(\vec{\mathrm{k}}_0)\,\delta^2(\vec{p} - \vec{\mathrm{k}}_0)$, $\qquad A_-(\vec{p}) = 0$

$\Rightarrow$ $\boxed{\mathfrak{f}(\mathbf{n}_0, \mathbf{n}) \text{ is given by } A_+ \text{ and } B_-.}$

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = \widehat{\mathbf{M}} \begin{bmatrix} A_- \\ B_- \end{bmatrix}$$

Solve for A_+ and B_-.

$$\mathfrak{f}(\mathbf{n}_0, \mathbf{n}) = \frac{1}{2\pi i k} \times \begin{cases} T^l(\mathbf{n}_0, \mathbf{n}) + 2\pi i k \Delta(\vec{\mathrm{n}}, \vec{\mathrm{n}}_0) & \text{for} \quad \mathrm{n}_{0x}, \mathrm{n}_x > 0 \\ R^l(\mathbf{n}_0, \mathbf{n}) & \text{for} \quad \mathrm{n}_{0x} > 0 > \mathrm{n}_x \\ R^r(\mathbf{n}_0, \mathbf{n}) & \text{for} \quad \mathrm{n}_{0x} < 0 < \mathrm{n}_x \\ T^r(\mathbf{n}_0, \mathbf{n}) + 2\pi i k \Delta(\vec{\mathrm{n}}, \vec{\mathrm{n}}_0) & \text{for} \quad \mathrm{n}_{0x}, \mathrm{n}_x < 0 \end{cases}$$

$$\Delta(\vec{\mathrm{n}}_0, \vec{\mathrm{n}}) := 2\pi i\, k^{-1} |\mathrm{n}_{0x}|\, \delta^2(\vec{\mathrm{n}} - \vec{\mathrm{n}}_0)$$

$$\psi(\mathbf{r}) \to \frac{1}{(2\pi)^{7/2}} \int_{\mathscr{D}_k} \frac{d\vec{p}}{\varpi(\vec{p})} e^{i\vec{p}\cdot\vec{r}} \left[A_\pm(\vec{p})\, e^{i\varpi(\vec{p})x} + B_\pm(\vec{p})\, e^{-i\varpi(\vec{p})x}\right] \quad \text{for} \quad x \to \pm\infty$$

Scattering solutions: $\psi(\mathbf{r}) \to \frac{N}{(2\pi)^{\frac{3}{2}}}\left[e^{i\mathbf{k}_0\cdot\mathbf{r}} + \frac{e^{ikr}}{r}\mathfrak{f}(\mathbf{n}_0,\mathbf{n})\right] \quad \text{for} \quad r \to \infty$

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$$\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) = \frac{1}{2\pi ik} \times \begin{cases} T^l(\mathbf{n}_0,\mathbf{n}) + 2\pi ik\Delta(\vec{\mathrm{n}},\vec{\mathrm{n}}_0) & \text{for} \quad \mathrm{n}_{0x},\mathrm{n}_x > 0 \\ R^l(\mathbf{n}_0,\mathbf{n}) & \text{for} \quad \mathrm{n}_{0x} > 0 > \mathrm{n}_x \\ R^r(\mathbf{n}_0,\mathbf{n}) & \text{for} \quad \mathrm{n}_{0x} < 0 < \mathrm{n}_x \\ T^r(\mathbf{n}_0,\mathbf{n}) + 2\pi ik\Delta(\vec{\mathrm{n}},\vec{\mathrm{n}}_0) & \text{for} \quad \mathrm{n}_{0x},\mathrm{n}_x < 0 \end{cases}$$

$\boxed{R^{l/r} \text{ and } T^{l/r} \text{ can be expressed in terms of } \widehat{\mathbf{M}}.}$

$$\Rightarrow \quad \boxed{\begin{aligned}
R^l(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,-\widehat{M}_{22}^{-1}\widehat{M}_{21}|\vec{\mathsf{k}}_0\rangle \\
T^l(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle \\
R^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{12}\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle \\
T^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle
\end{aligned}}$$

$$\widehat{M}_{ij} : \mathscr{F}_k^1 \to \mathscr{F}_k^1$$

$$\mathfrak{f}(\mathbf{n}_0,\mathbf{n}) = \frac{1}{2\pi i k}\times\begin{cases}
T^l(\mathbf{n}_0,\mathbf{n}) + 2\pi i k\Delta(\vec{\mathsf{n}},\vec{\mathsf{n}}_0) & \text{for} \quad \mathsf{n}_{0x},\mathsf{n}_x > 0 \\
R^l(\mathbf{n}_0,\mathbf{n}) & \text{for} \quad \mathsf{n}_{0x} > 0 > \mathsf{n}_x \\
R^r(\mathbf{n}_0,\mathbf{n}) & \text{for} \quad \mathsf{n}_{0x} < 0 < \mathsf{n}_x \\
T^r(\mathbf{n}_0,\mathbf{n}) + 2\pi i k\Delta(\vec{\mathsf{n}},\vec{\mathsf{n}}_0) & \text{for} \quad \mathsf{n}_{0x},\mathsf{n}_x < 0
\end{cases}$$

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\end{aligned}}$$

$$\widehat{M}_{ij}:\mathscr{F}_k^1\to\mathscr{F}_k^1$$

Compare with 1D:

$$\boxed{R^l=-\frac{M_{21}}{M_{22}},\qquad T^l=\frac{\det\mathbf{M}}{M_{22}},\qquad R^r=\frac{M_{12}}{M_{22}},\qquad T^r=\frac{1}{M_{22}}}$$

$$\mathfrak{f}(\mathbf{n}_0,\mathbf{n})=\frac{1}{2\pi ik}\times\begin{cases}
T^l(\mathbf{n}_0,\mathbf{n})+2\pi ik\Delta(\vec{\mathsf{n}},\vec{\mathsf{n}}_0) & \text{for}\quad \mathsf{n}_{0x},\mathsf{n}_x>0\\
R^l(\mathbf{n}_0,\mathbf{n}) & \text{for}\quad \mathsf{n}_{0x}>0>\mathsf{n}_x\\
R^r(\mathbf{n}_0,\mathbf{n}) & \text{for}\quad \mathsf{n}_{0x}<0<\mathsf{n}_x\\
T^r(\mathbf{n}_0,\mathbf{n})+2\pi ik\Delta(\vec{\mathsf{n}},\vec{\mathsf{n}}_0) & \text{for}\quad \mathsf{n}_{0x},\mathsf{n}_x<0
\end{cases}$$

Identify: $\mathscr{F}^1$ with $L^2(\mathbb{R}^2)$, and $\mathscr{F}_k^1$ with $L^2(\mathscr{D}_k)$

$$\Rightarrow \mathscr{F}^m \to L^2(\mathbb{R}^2) \otimes \mathbb{C}^{m\times 1}, \text{ and } \mathscr{F}_k^m \to L^2(\mathscr{D}_k) \otimes \mathbb{C}^{m\times 1}$$

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Theorem: Let $\forall x \in \mathbb{R}$, $\widehat{\mathrm{H}}(x) : \mathscr{F}^2 \to \mathscr{F}^2$ be the linear:

$$\boxed{\widehat{\mathrm{H}}(x) := \frac{1}{2} e^{-ix\widehat{\varpi}_r\widehat{\boldsymbol{\sigma}}_3} \widehat{\mathscr{V}}(x)\, \widehat{\varpi}^{-1} \widehat{\mathcal{K}}\, e^{ix\widehat{\varpi}_r\widehat{\boldsymbol{\sigma}}_3} - i\widehat{\varpi}_i\widehat{\boldsymbol{\sigma}}_3}$$

where

$$(\widehat{\varpi}\,\mathrm{f})(\vec{p}) := \varpi(\vec{p})\,\mathrm{f}(\vec{p}), \qquad \varpi(\vec{p}) := \sqrt{k^2 - \vec{p}^2}$$

$$(\widehat{\varpi}_r\,\mathrm{f})(\vec{p}) := \mathrm{Re}[\varpi(\vec{p})]\,\mathrm{f}(\vec{p}), \qquad (\widehat{\varpi}_i\,\mathrm{f})(\vec{p}) := \mathrm{Im}[\varpi(\vec{p})]\,\mathrm{f}(\vec{p})$$

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$$\big(\widehat{\mathscr{V}}(x)\,\mathbf{f}\big)(\vec{p}) := \big(v(x,i\vec{\nabla})\mathbf{f}\big)(\vec{p}) = \frac{1}{4\pi^2}\int_{\mathbb{R}^2} d\vec{q}\ \tilde{v}(x,\vec{p}-\vec{q})\,\mathbf{f}(\vec{q})$$

$$\tilde{v}(x,\vec{p}) := \int_{\mathbb{R}^2} d\vec{r}\,e^{-i\vec{p}\cdot\vec{r}}v(x,\vec{r})$$

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$$(\widehat{\boldsymbol{\sigma}}_3\,\mathrm{f})(\vec{p}) := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathrm{f}(\vec{p}), \qquad (\widehat{\mathcal{K}}\,\mathrm{f})(\vec{p}) := \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\mathrm{f}(\vec{p}).$$

Identify: $\mathscr{F}^1$ with $L^2(\mathbb{R}^2)$, and $\mathscr{F}_k^1$ with $L^2(\mathscr{D}_k)$

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Then,

$$\boxed{\widehat{\mathrm{M}} = \mathscr{T} \exp\left[-i\int_{-\infty}^{\infty} dx\; \widehat{\mathrm{H}}(x)\right]}$$

$\mathscr{H}$: A Hilbert space

$\widehat{X} : \mathscr{H} \to \mathscr{H}$ an invertible linear or antilinear operator

Definition: A linear operator $\widehat{L} : \mathscr{H} \to \mathscr{H}$ is called

– $\widehat{X}$-pseudo-Hermitian if $\widehat{L}^\dagger = \widehat{X}\,\widehat{L}\,\widehat{X}^{-1}$

– $\widehat{X}$-pseudo-anti-Hermitian if $\widehat{L}^\dagger = -\widehat{X}\,\widehat{L}\,\widehat{X}^{-1}$

– $\widehat{X}$-pseudo-unitary if $\widehat{L}^\dagger = \widehat{X}\,\widehat{L}^{-1}\widehat{X}^{-1}$ & $\widehat{X}$ is linear

– $\widehat{X}$-anti-pseudo-unitary if $\widehat{L}^\dagger = \widehat{X}\,\widehat{L}^{-1}\widehat{X}^{-1}$ & $\widehat{X}$ is antilinear

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Theorem: Let $\mathscr{H}$ be the Hilbert space of a quantum system with Hamitonian $\widehat{H}(t)$, and $\widehat{\mathcal{X}}:\mathscr{H}\to\mathscr{H}$ be an invertible antilinear operator. Then $\widehat{\mathcal{X}}$-pseudo-anti-Hermiticity of $\widehat{H}(t)$ implies $\widehat{\mathcal{X}}$-anti-pseudo-unitarity of its evolution operator $\widehat{U}(t,t_0)$, i.e.,

$$\widehat{H}(t)^\dagger=-\widehat{X}\,\widehat{H}(t)\widehat{X}^{-1}\quad\Rightarrow\quad\widehat{U}(t,t_0)^\dagger=\widehat{X}\,\widehat{U}(t,t_0)^{-1}\widehat{X}^{-1}$$

Theorem: Let $\widehat{\mathcal{P}}, \widehat{\mathcal{T}}, \widehat{\mathfrak{T}} : \mathscr{F}^m \to \mathscr{F}^m$ and $\widehat{\mathbf{\Omega}} : \mathscr{F}^2 \to \mathscr{F}^2$ be operators defined by

$$(\widehat{\mathcal{P}}\mathbf{f})(\vec{p}) := \mathbf{f}(-\vec{p}), \qquad (\widehat{\mathcal{T}}\mathbf{f})(\vec{p}) := \mathbf{f}(\vec{p})^*,$$

$$\widehat{\mathfrak{T}} := \widehat{\varpi}^{-1}\widehat{\mathcal{P}}\widehat{\mathcal{T}}, \qquad (\widehat{\mathbf{\Omega}}\mathbf{f})(\vec{p}) := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{f}(\vec{p}),$$

and $v : \mathbb{R}^3 \to \mathbb{C}$ be a short-range potential.

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and $v:\mathbb{R}^3\to\mathbb{C}$ be a short-range potential. Then the fundamental transfer matrix of v is $\widehat{\mathbf{\Omega}}\widehat{\mathfrak{T}}$-anti-pseudo-unitary, i.e.,

$$\boxed{\widehat{\mathbf{M}}^\dagger=(\widehat{\mathbf{\Omega}}\widehat{\mathfrak{T}})\,\widehat{\mathbf{M}}^{-1}(\widehat{\mathbf{\Omega}}\widehat{\mathfrak{T}})^{-1}}\ .$$

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Proof: $\widehat{\mathbf{M}}=\mathscr{T}\exp\left[-i\int_{-\infty}^{\infty}dx\;\widehat{\mathbf{H}}(x)\right]$ and $\widehat{\mathbf{H}}(x)$ is $\widehat{\mathbf{\Omega}}\widehat{\mathfrak{T}}$-pseudo-anti-Hermitian.

$$\widehat{\mathbf{H}}(x):=\frac{1}{2}e^{-ix\widehat{\varpi}_r\widehat{\boldsymbol{\sigma}}_3}\widehat{\mathscr{V}}(x)\,\widehat{\varpi}^{-1}\widehat{\mathcal{K}}\,e^{ix\widehat{\varpi}_r\widehat{\boldsymbol{\sigma}}_3}-i\widehat{\varpi}_i\widehat{\boldsymbol{\sigma}}_3$$

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Compare with 1D:

$$\boxed{\mathbf{M}^\dagger=\mathbf{\Omega}\mathcal{T}\,\mathbf{M}^{-1}(\mathbf{\Omega}\mathcal{T})^{-1}}$$

Proof of Reciprocity Theorem in 3D:

$$R^l(\mathbf{n}_0,\mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|-\widehat{M}_{22}^{-1}\widehat{M}_{21}|\vec{\mathsf{k}}_0\rangle$$

$$T^l(\mathbf{n}_0,\mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle$$

$$R^r(\mathbf{n}_0,\mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{12}\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle$$

$$T^r(\mathbf{n}_0,\mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle$$

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Reciprocity $\Leftrightarrow$

$$\boxed{\begin{aligned} R^l(\mathbf{n}_0,\mathbf{n}) &= R^l(-\mathbf{n},-\mathbf{n}_0) \\ R^r(\mathbf{n}_0,\mathbf{n}) &= R^r(-\mathbf{n},-\mathbf{n}_0) \\ T^l(\mathbf{n}_0,\mathbf{n}) &= T^r(-\mathbf{n},-\mathbf{n}_0) \\ T^r(\mathbf{n}_0,\mathbf{n}) &= T^l(-\mathbf{n},-\mathbf{n}_0) \end{aligned}}$$

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T^l(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle\\
R^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{12}\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle\\
T^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle
\end{aligned}
}
$$

Reciprocity $\Leftrightarrow$

$$
\boxed{
\begin{aligned}
R^l(\mathbf{n}_0,\mathbf{n}) &= R^l(-\mathbf{n},-\mathbf{n}_0)\\
R^r(\mathbf{n}_0,\mathbf{n}) &= R^r(-\mathbf{n},-\mathbf{n}_0)\\
T^l(\mathbf{n}_0,\mathbf{n}) &= T^r(-\mathbf{n},-\mathbf{n}_0)\\
T^r(\mathbf{n}_0,\mathbf{n}) &= T^l(-\mathbf{n},-\mathbf{n}_0)
\end{aligned}
}
$$

$$\Updownarrow$$

$$
\boxed{
\begin{aligned}
(\widehat{M}_{22}^{-1}\widehat{M}_{21})^\dagger &= \widehat{\mathfrak{T}}\,(\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1}\\
(\widehat{M}_{12}\widehat{M}_{22}^{-1})^\dagger &= \widehat{\mathfrak{T}}\,(\widehat{M}_{12}\widehat{M}_{22}^{-1})\,\widehat{\mathfrak{T}}^{-1}\\
\widehat{M}_{22}^{-1\dagger} &= \widehat{\mathfrak{T}}\,(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1}
\end{aligned}
}
$$

Proof of Reciprocity Theorem in 3D:

$$\boxed{\begin{aligned} R^l(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|-\widehat{M}_{22}^{-1}\widehat{M}_{21}|\vec{\mathsf{k}}_0\rangle\\ T^l(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle\\ R^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{12}\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle\\ T^r(\mathbf{n}_0,\mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle \end{aligned}}$$

Reciprocity $\Leftrightarrow$

$$\boxed{\begin{aligned} R^l(\mathbf{n}_0,\mathbf{n}) &= R^l(-\mathbf{n},-\mathbf{n}_0)\\ R^r(\mathbf{n}_0,\mathbf{n}) &= R^r(-\mathbf{n},-\mathbf{n}_0)\\ T^l(\mathbf{n}_0,\mathbf{n}) &= T^r(-\mathbf{n},-\mathbf{n}_0)\\ T^r(\mathbf{n}_0,\mathbf{n}) &= T^l(-\mathbf{n},-\mathbf{n}_0) \end{aligned}}$$

$$\Updownarrow$$

$$\boxed{\widehat{\mathbf{M}}^\dagger = (\widehat{\boldsymbol{\Omega}}\widehat{\mathfrak{T}})\,\widehat{\mathbf{M}}^{-1}(\widehat{\boldsymbol{\Omega}}\widehat{\mathfrak{T}})^{-1}} \quad \Leftrightarrow \quad \boxed{\begin{aligned} (\widehat{M}_{22}^{-1}\widehat{M}_{21})^\dagger &= \widehat{\mathfrak{T}}\,(\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1}\\ (\widehat{M}_{12}\widehat{M}_{22}^{-1})^\dagger &= \widehat{\mathfrak{T}}\,(\widehat{M}_{12}\widehat{M}_{22}^{-1})\,\widehat{\mathfrak{T}}^{-1}\\ \widehat{M}_{22}^{-1\dagger} &= \widehat{\mathfrak{T}}\,(\widehat{M}_{11}-\widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1} \end{aligned}}$$

$$\boxed{\widehat{\mathbf{M}}^{\dagger} = (\widehat{\boldsymbol{\Omega}}\widehat{\mathfrak{T}})\,\widehat{\mathbf{M}}^{-1}(\widehat{\boldsymbol{\Omega}}\widehat{\mathfrak{T}})^{-1}} \quad \Leftrightarrow \quad \boxed{\begin{array}{l} (\widehat{M}_{22}^{-1}\widehat{M}_{21})^{\dagger} = \widehat{\mathfrak{T}}\,(\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1} \\ (\widehat{M}_{12}\widehat{M}_{22}^{-1})^{\dagger} = \widehat{\mathfrak{T}}\,(\widehat{M}_{12}\widehat{M}_{22}^{-1})\,\widehat{\mathfrak{T}}^{-1} \\ \widehat{M}_{22}^{-1\dagger} = \widehat{\mathfrak{T}}\,(\widehat{M}_{11} - \widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})\,\widehat{\mathfrak{T}}^{-1} \end{array}}$$

$$T^{l}(\mathbf{n}_0, \mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11} - \widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle$$

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$$\begin{aligned} T^l(\mathbf{n}_0, \mathbf{n}) &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|(\widehat{M}_{11} - \widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21})|\vec{\mathsf{k}}_0\rangle \\ &= 4\pi^2 k\,\varpi(\vec{\mathsf{k}})\langle-\vec{\mathsf{k}}_0|\widehat{M}_{22}^{-1}|-\vec{\mathsf{k}}\rangle \end{aligned}$$

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Compare with 1D:

$$T^{l} = T^{r} = \frac{1}{M_{22}}$$

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$$T^{r}(\mathbf{n}_0,\mathbf{n}) = 4\pi^2 k\,\varpi(\vec{\mathsf{k}}_0)\,\langle\vec{\mathsf{k}}|\,\widehat{M}_{22}^{-1}|\vec{\mathsf{k}}_0\rangle$$

$$\widehat{M}_{11}\widehat{M}_{22} - \widehat{M}_{12}\widehat{M}_{22}^{-1}\widehat{M}_{21}\widehat{M}_{22} = \widehat{\mathfrak{T}}^{-1}\widehat{M}_{22}^{-1\dagger}\,\widehat{\mathfrak{T}}\,\widehat{M}_{22}$$

Compare with 1D:

$$T^{l} = T^{r} = \frac{1}{M_{22}} \qquad\qquad \det\mathbf{M} = 1$$

Concluding Remarks

- **Scattering** phenomenon may be understood in terms of the dynamics of a **non-unitary quantum system**.

$$\boxed{\widehat{\mathrm{M}} = \mathscr{T} \exp\left[-i\int_{-\infty}^{\infty} dx\ \widehat{\mathrm{H}}(x)\right]}$$

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- This has opened up a **new route** for

- finding new exactly solvable scattering problems

- treating point scatterers in 2D and 3D,

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- This has opened up a **new route** for

- finding new exactly solvable scattering problems

- treating point scatterers in 2D and 3D,

- realization of broadband directional invisibility

- obtaining analytic results in low-frequency scattering

- constructing the first examples of potentials for which the N-th order Born approximation is exact

- understanding the **origin of reciprocity**

$$\boxed{\widehat{\mathrm{M}}^{\dagger} = (\widehat{\Omega}\widehat{\mathfrak{T}})\,\widehat{\mathrm{M}}^{-1}(\widehat{\Omega}\widehat{\mathfrak{T}})^{-1}}$$

• As it stands this approach is the **only basic alternative** to the textbook treatment of stationary scattering.

• There are many related **open problems awaiting** the attention of **physicists** as well as **mathematicians**.

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References:

- arXiv: 2109.06528 [Phys Rev A **104**, 032222 (2021)]

- arXiv: 2204.09554 [Ann Phys **443**, 168966 (2022)]

- arXiv: 2308.03689 [Appl Phys Lett **123**, 191104 (2023)]

- arXiv: 2407.19983 [J Phys A **57**, 335205 (2024)]

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Thank you for your attention.